3. INTRODUCTION TO STRUCTURAL DYNAMICS

The transient dynamic equilibrium equation for a linear discrete structure (equations of motion):

$$\underset{n\times n}{[M]}\underset{n\times 1}{\{\overset{\circ\circ}{q}\}}+\underset{n\times n}{[C]}\underset{n\times 1}{\{\overset{\circ}{q}\}}+\underset{n\times n}{[K]}\underset{n\times 1}{\{q\}}=\underset{n\times 1}{\{F(t)\}},$$

$$\{\overset{\circ}{q}\}=\frac{d}{dt}\{q(t)\}=\begin{Bmatrix}\dfrac{dq_1(t)}{dt}\\ \dfrac{dq_2(t)}{dt}\\ \dots \\ \dfrac{dq_n(t)}{dt}\end{Bmatrix}, \qquad \{\overset{\circ\circ}{q}\}=\frac{d^2}{dt^2}\{q(t)\}=\begin{Bmatrix}\dfrac{d^2q_1(t)}{dt^2}\\ \dfrac{d^2q_2(t)}{dt^2}\\ \dots \\ \dfrac{d^2q_n(t)}{dt^2}\end{Bmatrix}$$

where:

[M] = structural mass matrix [C] = structural damping matrix
[K] = structural stiffness matrix
{q} = nodal displacement vector , {q'} = nodal velocity vector
{q''} = nodal acceleration vector
{F(t)} = applied load vector

$$U=\frac{1}{2}\lfloor q\rfloor[K]\{q\}. \qquad T=\frac{1}{2}\lfloor \overset{\circ}{q}\rfloor[M]\{\overset{\circ}{q}\}$$

Rayleigh model of damping

$$[C]=\alpha_t[M]+\beta_t[K],$$

α_t β_t - external and internal damping coefficients

Special cases :

{F(t)} = {0} - free vibrations

{F(t)} = {0} [C]=[0] - free undamped vibrations (natural v.)

[M]=[0], {F(t)} = {0} [C]=[0] - linear static problem

NATURAL VIBRATIONS - modal analysis

$$[M]\{\ddot{q}\}+[K]\{q\}=\{0\}.$$ Second order set of differential equations

General solution $\{q(t)\}=\{q\}_A\cos\omega t+\{q\}_B\sin\omega t$

$\{q\}_A$ i $\{q\}_B$ - vectors evaluated from the initial conditions, ω - natural circular frequency

$$\{\ddot{q}\}=-\omega^2\{q\}_A\cos\omega t-\omega^2\{q\}_B\sin\omega t=-\omega^2\{q\}.$$

$$-\omega^2[M]\{q\}+[K]\{q\}=\{0\}, \qquad [K]\{q\}=\omega^2[M]\{q\}$$

Obtained the **eigenvalue problem** : $$([K]-\omega^2[M])\{q\}=\{0\}.$$

Trivial solution $$\{q\}=\{0\}$$

Nontrivial solutions $$\det([K]-\omega^2[M])=0.$$

The determinant - polynomial of n-th degree in terms of ω^2 . The solutions ω_i are the natural frequencies (eigenvalues). The corresponding eigenvectors $\{q\}_i$ are called the natural modes.

They can be scaled (if $\{q\}_i$ is the eigenvector then $\alpha\{q\}_i$ also satisfies the eigenvalue problem)

Usually the eigenvectors are normalized , $\lfloor q\rfloor_i\{q\}_j=\lfloor q\rfloor_i[I]\{q\}_j=\delta_{ij}$ or $\lfloor q\rfloor_i[M]\{q\}_j=\delta_{ij}$.

The solution is much more time-consuming than the solution of a set of linear equations in static analysis.

Iterative numerical techniques are used to find the limited number of eigenvalues (natural frequencies) within the interesting range..

EIGENVALUES AND EIGENVECTORS IN ALGEBRA

Consider the special form of the linear system in which the right-hand side vector **y** is a multiple of the solution vector **x**:

$$\mathbf{Ax} = \lambda\, \mathbf{x}$$

or, written in full,

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & \lambda x_1 \\ a_{21}x_2 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & \lambda x_2 \\ \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & \lambda x_n \end{array}$$

This is called the standard (or classical) ***algebraic eigenproblem***. The system can be rearranged into the homogeneous form

$$(\mathbf{A} - \lambda\mathbf{I})\, \mathbf{x} = \mathbf{0}.$$

A nontrivial solution of this equation is possible if and only if the coefficient matrix $\mathbf{A}-\lambda\mathbf{I}$ is singular.

Such a condition can be expressed as the vanishing of the determinant : $|\mathbf{A} - \lambda\mathbf{I}| = 0$

When this determinant is expanded, we obtain an algebraic polynomial equation in λ of degree n:

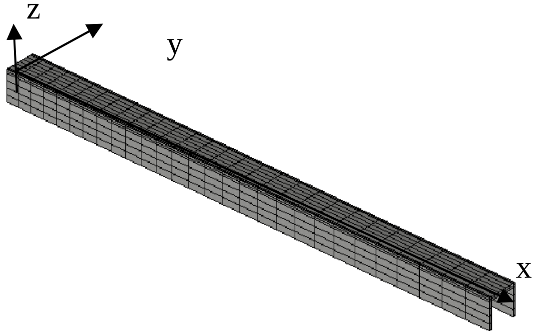
$$P(\lambda) = \lambda^n + \alpha_1\lambda^{n-1} + \cdot \quad \cdot \quad \cdot + \alpha_n = 0$$

This is known as the ***characteristic equation*** of the matrix **A**. The left-hand side is called the ***characteristic polynomial***. We known that a polynomial of degree n has n (generally complex) roots $\lambda_1, \lambda_2, \ldots, \lambda_n$. These n numbers are called the ***eigenvalues, eigenroots*** or ***characteristic values*** of matrix **A**.

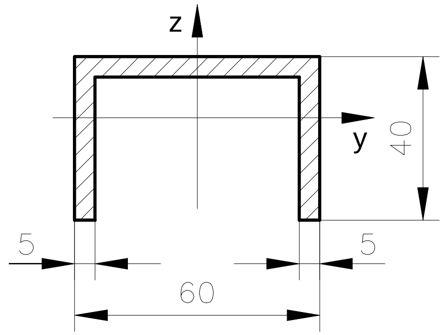
With each eigenvalue λi there is an associated vector $\mathbf{x}_i$ that satisfies $\mathbf{Ax}_i = \lambda_i\, \mathbf{x}_i.$

This $\mathbf{x}i$ is called an ***eigenvector* or *characteristic vector***. An eigenvector is unique only up to a scale factor: if $\mathbf{x}_i$ is an eigenvector, so is $\beta\mathbf{x}_i$ where β is an arbitrary nonzero number. Eigenvectors are often *normalized* so that e.g. their Euclidean length is 1.

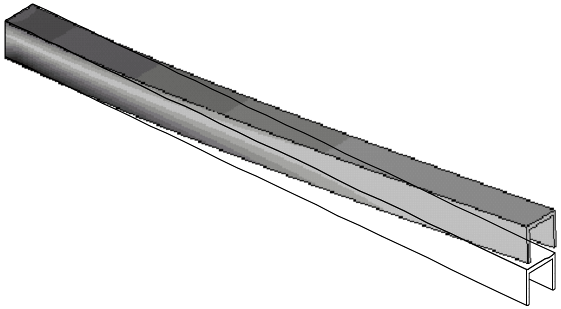
Example – free vibrations of the cantilever beam - mode shapes and natural frequencies



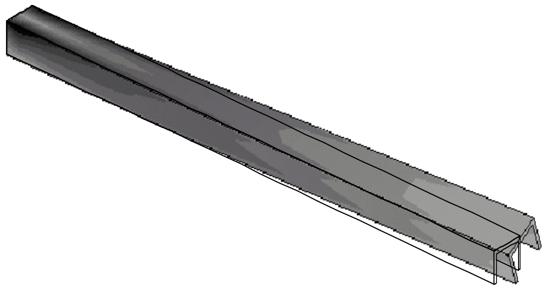
FE model



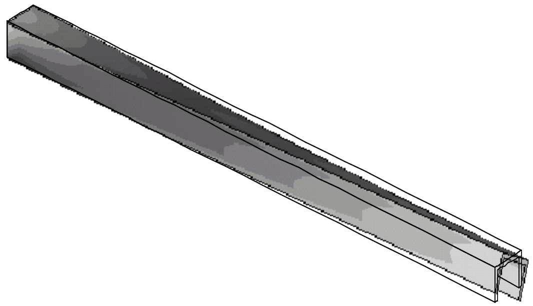
$E=2*10^5$ MPa
$\nu=0.3$
$\rho=8*10^3 \frac{kg}{m^3}$



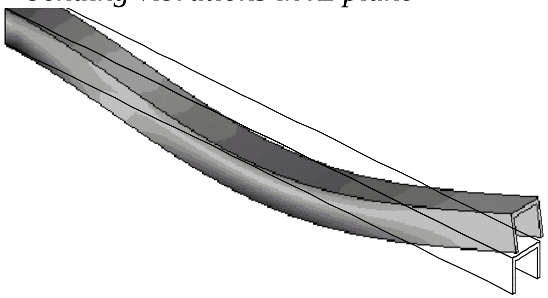
mode shape 1 , $\omega_1=219.1$
bending vibrations in xz plane



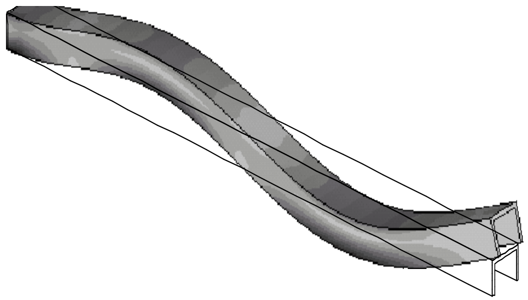
mode shape 2 , $\omega_2=333.3$
torsional vibrations



mode shape 3 , $\omega_3=769.5$,
Bending and torsional vibrations

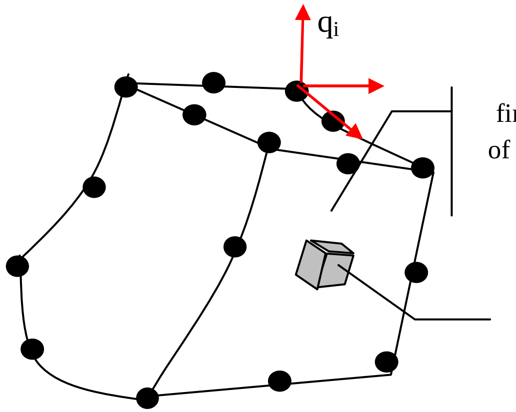


mode shape 4 $\omega_4=1353.73$



mode shape 7 , $\omega_1=3704.3$, bending

Mass matrix of a finite element



Defines kinetic energy of an element $T_e = \frac{1}{2}\lfloor \dot{q} \rfloor_e [m]_e \{\dot{q}\}_e$.

Displacement vector within the element

$$\{u\} = [N]\{q\}_e ,$$

Velocity vector

$$\frac{d}{dt}\{u\} = [N]\{\dot{q}\}_e .$$

Kinetic energy of the part $d\Omega_e$ of the finite element Ω_e

$$dT_e = \frac{1}{2}\lfloor \dot{u} \rfloor dm \{\dot{u}\} = \frac{1}{2}\lfloor \dot{u} \rfloor \rho \{\dot{u}\} d\Omega_e ,$$

ρ - density

$$dT_e = \frac{1}{2}\lfloor \dot{q} \rfloor_e [N]^T \cdot \rho [N] \{\dot{q}\}_e \, d\Omega_e .$$

$$T_e = \frac{1}{2}\lfloor \dot{q} \rfloor_e \int_{\Omega_e} [N]^T \rho [N] d\Omega_e \{\dot{q}\}_e .$$

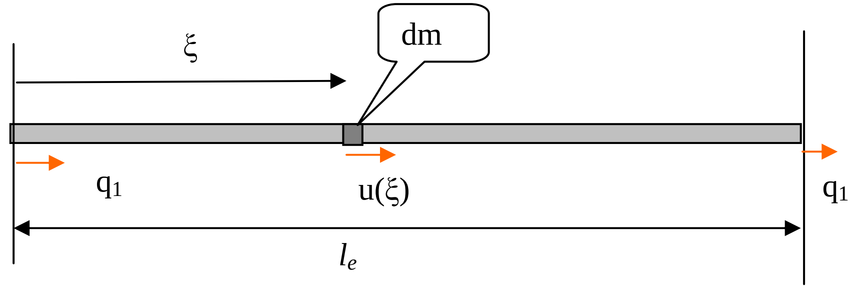
General formula for the consistent mass matrix of a finite element

$$T_e = \frac{1}{2}\lfloor \dot{q} \rfloor_e [m]_e \{\dot{q}\}_e \qquad [m]_e = \int_{\Omega_e} [N]^T \rho [N] d\Omega_e ,$$

.

The relation describes so named consistent mass matrix (determined using the same approach as for the stiffness matrix).
To simplify the calculations it can be used also so called lumped mass matrix (diagonal)

FEM relations for rod element



The mass matrix for an axial member

$$T_e = \int_0^{l_e} \frac{dm(\dot{u})^2}{2} = \frac{1}{2}\int_0^{l_e} (\dot{u})^2 \rho A d\xi .$$

Velocity of the particles along the element

$$\dot{u}(\xi) = \lfloor N_1(\xi), N_2(\xi) \rfloor \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix},$$

$$N_1 = 1 - \frac{\xi}{l_e}, \qquad N_2 = \frac{\xi}{l_e},$$

ξ- local coordinate

Evaluating the integrals we get

$$T_e = \frac{1}{2}\lfloor \dot{q}_1, \dot{q}_2 \rfloor [m]_e \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$$

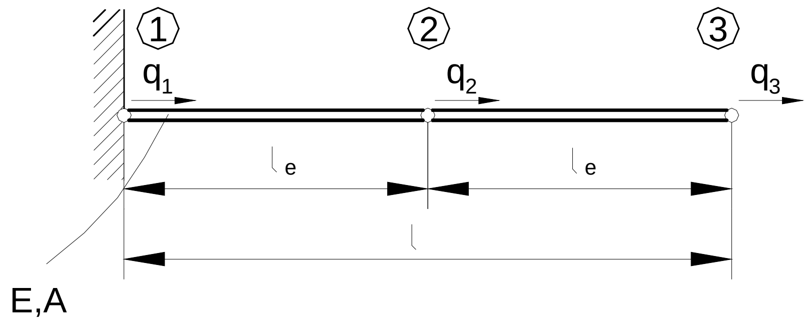
$$[m]_e = \int_{\Omega_e} \{N\}\,\rho \lfloor N \rfloor d\Omega_e$$

$$[m_e] = \frac{\rho A l_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The diagonal (lumped) form of the matrix

$$[m]_e = \begin{bmatrix} \dfrac{\rho A l_e}{2} & 0 \\ 0 & \dfrac{\rho A l_e}{2} \end{bmatrix}.$$

Example



Free vibrations of the rod fixed at one end – FE model with 2 elements

<u>The analytical solution</u>

: $$\omega_i^s = \frac{2i-1}{2}\pi\frac{1}{l}\sqrt{\frac{E}{\rho}}.$$

$$\omega_1^s = 1.5708\frac{1}{l}\sqrt{\frac{E}{\rho}},$$

$$\omega_2^s = 4.7124\frac{1}{l}\sqrt{\frac{E}{\rho}},$$

<u>FE solution using model with 2 finite elements</u>

Free vibrations equation

$$\left([K]-\omega^2[M]\right)\{q\}=\{0\},$$

The stiffness matrix and the mass matrix for both elements;

$$[k]_e = \frac{EA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad [m]_e = \frac{\rho A l_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad l_e = \frac{l}{2}.$$

$$\left(\frac{EA}{l_e}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix} - \omega^2\frac{\rho A l_e}{6}\begin{bmatrix}2 & 1 & 0\\ 1 & 4 & 1\\ 0 & 1 & 2\end{bmatrix}\right)\begin{Bmatrix}q_1\\ q_2\\ q_3\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix}.$$

$q_1 = 0$

$$\left(\frac{EA}{l_e}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix} - \omega^2\frac{\rho A l_e}{6}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}\right)\begin{Bmatrix}q_2\\ q_3\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}.$$

Substituting $\lambda = \omega^2 \frac{\rho A l_e}{6} \Big/ \frac{EA}{l_e} = \frac{\rho l_e^2}{6E}\omega^2$

we have

$$\det\begin{bmatrix}2-4\lambda & -(1+\lambda)\\ -(1+\lambda) & (1-2\lambda)\end{bmatrix} = 0.$$

Ihe roots of the characteristic equation of the matrix:
$\lambda_1 = 0.1082$, $\lambda_2 = 1.3204$ and consequently

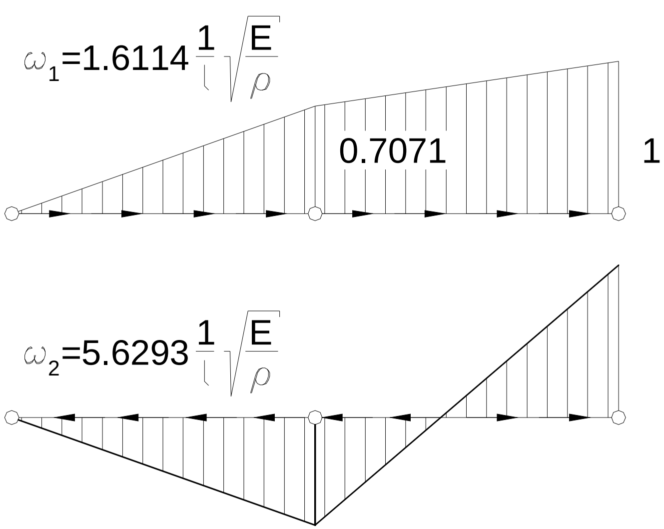
$$\omega_1 = 0.8057\frac{1}{l_e}\sqrt{\frac{E}{\rho}} = 1.6114\frac{1}{l}\sqrt{\frac{E}{\rho}},$$

$$\omega_2 = 2.8148\frac{1}{l_e}\sqrt{\frac{E}{\rho}} = 5.6293\frac{1}{l}\sqrt{\frac{E}{\rho}}.$$

Comparing with the exact solution

we have the relative errors of the natural frequencies 2.6% and 19.5%.

<u>Mode shapes (eigenvectors</u>



Assuming $q_3 = \Delta$ and $q_1 = 0$.

$$\lfloor q \rfloor_1 = \lfloor 0, \quad 0.707\Delta, \quad \Delta \rfloor,$$

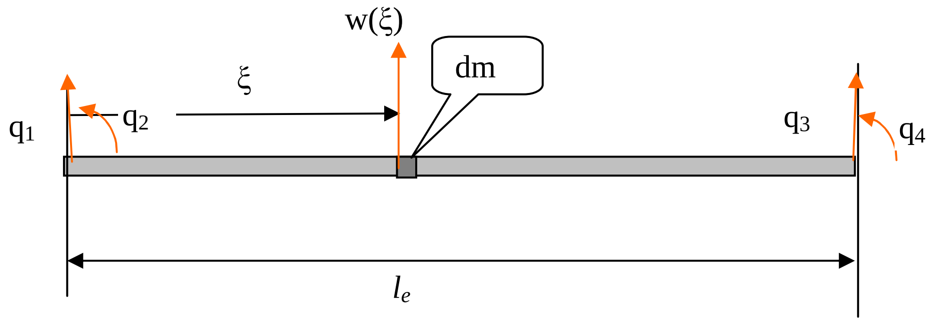
$$\lfloor q \rfloor_2 = \lfloor 0, \; -0.707\Delta, \quad \Delta \rfloor.$$

The mass matrix of the simple beam element

Kinetic energy of the segment $d\xi$ of the beam

$$dT_e = dm \cdot (\dot{w})^2 / 2$$ (without the rotational movement)

Velocity of the segment



$$\dot{w}(\xi) = \lfloor N_1(\xi), N_2(\xi), N_3(\xi), N_4(\xi) \rfloor \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{Bmatrix},$$

N_i - shape functions of the beam element

Kinetic energy of the element :

$$T_e = \int_0^{l_e} dT_e = \frac{1}{2} \int_0^{l_e} (\dot{w})^2 \rho A d\xi .$$

$$T_e = \frac{1}{2} \lfloor \dot{q}_1, \dot{q}_2, \dot{q}_3, \dot{q}_4 \rfloor \cdot \frac{\rho A l_e}{420} \begin{bmatrix} 156 & 22l_e & 54 & -13l_e \\ & 4l_e^2 & 13l_e & -3l_e^2 \\ \text{sym.} & & 156 & -22l_e \\ & & & 4l_e^2 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{Bmatrix}$$

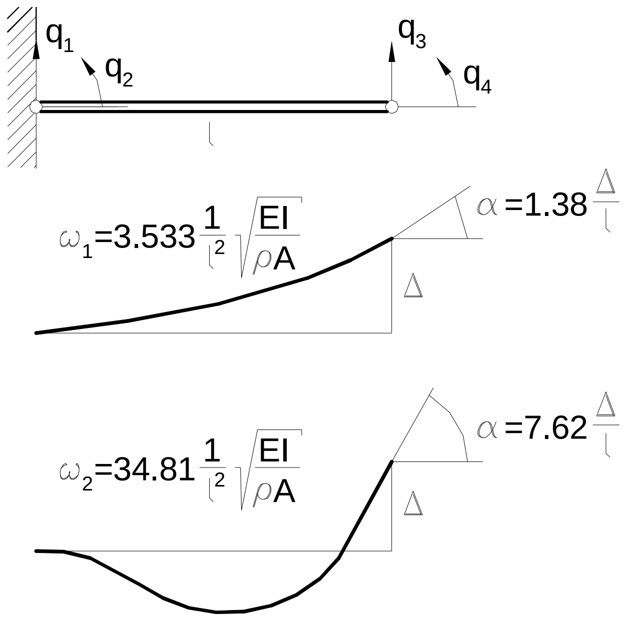
The mass matrix

$$[m]_e = \frac{\rho A l_e}{420} \begin{bmatrix} 156 & 22l_e & 54 & -13l_e \\ & 4l_e^2 & 13l_e & -3l_e^2 \\ & & 156 & -22l_e \\ & & & 4l_e^2 \end{bmatrix}$$

The same result may be derived using thegeneral formula $[m]_e = \int_{\Omega_e} [N]^T \rho [N] d\Omega_e$,

Example - Natural frequencies of the cantilever beam .

One element FE model



The exact analytical solution

$$\omega_1^s = 3.5156 \cdot \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}},$$

$$\omega_2^s = 22.0346 \cdot \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}},$$

$$\omega_i^s = \left[\frac{(2i-1)\pi}{2}\right]^2 \cdot \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}, \qquad i = 3, 4...,$$

FEM - the eigenvalue problem

$$\left(\frac{2EI}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ & 2l^2 & -3l & l^2 \\ & & 6 & -3l \\ & & & 2l^2 \end{bmatrix} - \omega^2 \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ & 4l^2 & 13l & -3l^2 \\ & & 156 & -22l \\ & & & 4l^2 \end{bmatrix}\right)\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}.$$

$q_1 = 0$, $q_2 = 0$

$$\left(\frac{2EI}{l^3}\begin{bmatrix} 6 & -3l \\ -3l & 2l^2 \end{bmatrix} - \frac{\omega^2 \rho A l}{420}\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix}\right)\begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\det\left(\begin{bmatrix} 6 & -3l \\ -3l & 2l^2 \end{bmatrix} - \lambda \begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix}\right) = 0,$$

Using the new parameter $\lambda = \dfrac{\rho A l^4}{840 EI} \cdot \omega^2$.

we obtain the characteristic equation

$$140\lambda^2 - 204\lambda + 3 = 0,$$

and the roots

$$\lambda_1 = 1.4857 \cdot 10^{-2},$$
$$\lambda_2 = 1.4423.$$

Thus

$$\omega_1 = 3,533 \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}, \qquad \omega_2 = 34,81 \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}.$$

Eigenvectors

$$\left[\begin{array}{c|c} 6-156\lambda & -3l+22l\lambda \\ \hline -3l+22l\lambda & 2l^2-4l^2\lambda \end{array}\right]\begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$

The vectors $\lfloor q \rfloor_1 = \lfloor q_3, q_4 \rfloor_1$ and $\lfloor q \rfloor_2 = \lfloor q_3, q_4 \rfloor_2$ corresponding to ω_1 i ω_2 (λ_1 i λ_2).

$$q_4 = \frac{-(6-156\lambda)}{(-3+22\lambda)}\cdot\frac{q_3}{l} \quad \textbf{\textit{or}} \quad q_4 = \frac{-3+22\lambda}{-2+4\lambda}\cdot\frac{q_3}{l}.$$

Assuming $q_3 = \Delta$ we obtain for the first mode $q_4 = 1.38\dfrac{\Delta}{l}$, and for the second mode $q_4 = 7,62\cdot\dfrac{\Delta}{l}$.

$$\lfloor q \rfloor_1 = \left\lfloor 0, 0, \Delta, 1.37\frac{\Delta}{l} \right\rfloor,$$

$$\lfloor q \rfloor_2 = \left\lfloor 0, 0, \Delta, 7.62\frac{\Delta}{l} \right\rfloor.$$

IN FEM MODAL ANALYSIS: Good accuracy of the results (frequencies, mode shapes) even for rough meshing.

Usually lower natural frequencies are determined with better accuracy

Summary- types of dynamic analyses in FEM:

Transient Dynamic

All dynamic analysis types in the ANSYS program are based on the following general equation of motion for a finite element system:

$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{F(t)\}$

where:

[M] mass matrix , [C] damping matrix , [K] stiffness matrix

$\{\ddot{u}\}$ nodal acceleration vector, $\{\dot{u}\}$ nodal velocity vector, {u} nodal displacement vector

{F} load vector , (t) time

Transient dynamic analysis (also known as time-history analysis) is used to determine the dynamic response of a structure subjected to time-dependent loads. There are three basic methods of a transient dynamic solution: full transient dynamic method, reduced method, and mode superposition..

The full transient dynamic is the most general. It has full nonlinear capability and may include plasticity, creep, large deflection, large strain, stress stiffening, and nonlinear elements.

Modal

Modal analysis is useful for any application in which the natural frequencies of a structure are of interest

For example, a machine component should be designed to produce natural frequencies that will prevent the component from vibrating at one of its fundamental modes under operating conditions.

Modal analysis is used to extract the natural frequencies and mode shapes of a structure. It is important as a first step to any dynamic analysis because knowledge of the structure's fundamental mode shapes and frequencies can help characterize its dynamic response. Some transient and harmonic solution procedures require the results of a modal analysis.

For undamped cases (which are most common for modal analysis) the damping term, $[C]\{\dot{u}\}$, is ignored and the equation reduces to:

$([K] - \omega^2[M])\{u\} = 0$

where ω^2 (the square of natural frequencies) represents the eigenvalues, and $\{u\}$ represents the eigenvectors (the mode shapes, which do not change with time).

Harmonic Response

Harmonic response analysis is used to determine the steady-state response of a linear structure to a sinusoidally varying forcing function. This analysis type is useful for studying the effects of load conditions that vary harmonically with time, such as those experienced by the housings, mountings, and foundations of rotating machinery.

Response Spectrum

A response spectrum analysis can be used to determine the response of a structure to shock loading conditions.

This analysis type uses the results of a modal analysis along with a known spectrum to calculate maximum displacements and stresses that occur in the structure at each of its natural frequencies. A typical response spectrum application is seismic analysis, which is used to study the effects of earthquakes on structures such as piping systems, towers, and bridges.

Random Vibration

Random vibration analysis is a type of spectrum analysis used to study the response of a structure to random excitations, such as those generated by jet or rocket engines.